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Geometric Finiteness and Non-quasinormal Modes of the BTZ Black Hole

Kumar S. Gupta[1]

Theory Division
Saha Institute of Nuclear Physics
1/AF Bidhannagar
Calcutta - 700064, India

Siddhartha Sen[2]

Hamilton Mathematics Institute TCD, Dublin 2, Ireland
and
Department of Theoretical Physics
Indian Association for the Cultivation of Science
Calcutta - 700032, India

Abstract

The BTZ black hole is geometrically finite. This means that its three dimensional hyperbolic structure as encoded in its metric is in 1-1 correspondence with the Teichmuller space of its boundary which is a two torus. The equivalence of different Teichmuller parameters related by the action of the modular group therefore requires the invariance of the monodromies of the solutions of the wave equation around the inner and outer horizons in the BTZ background. We show that this invariance condition leads to the non-quasinormal mode frequencies discussed by Birmingham and Carlip.

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[1]Email: kumars.gupta@saha.ac.in, Regular Associate, Abdus Salam ICTP, Trieste, Italy.
[2]Email: sen@maths.tcd.ie

Quasinormal modes associated with black hole metric perturbations [1] were introduced by Vishveshwara [2] and have been found to be a useful tool in analyzing the properties of a black hole using an external probe. The usual calculation of the quasinormal modes in a given geometry involves finding the solutions of the wave equation with suitable boundary conditions. In asymptotically flat space-times, the quasinormal modes are defined as the solutions which are purely ingoing at the horizon and purely outgoing at infinity [3]. However, for the case of the BTZ black hole [4] which is asymptotically AdS, the potential in the radial part of the corresponding wave equation diverges at infinity. In this case, the quasinormal modes are defined as solutions which are purely ingoing at the horizon and which vanish at infinity [5, 6].

Subsequently, the study of quasinormal modes was given a geometric formulation following the work of Motl and Neitzke [7]. The essential feature of this approach involves the extension of the wave equation beyond the physical region between the horizon and infinity by analytically continuing the radial variable r to the whole complex plane. For the Schwarzschild case, this leads to a differential equation with regular singular points at $r = 0,\ 1$(horizon) and ∞. The solutions of the wave equation with appropriate boundary conditions are multivalued around $r = 0$ and $r = 1$, leading to nontrivial monodromies around suitably chosen closed contours in the complex r plane. An equation relating the relevant monodromies then leads to the evaluation of the quasinormal modes. In the case of the BTZ black hole a similar analysis involving the monodromies was done by Musiri and Siopsis [8]. In this case, the solution of the wave equation vanishes at infinity, which again leads to an equation relating the monodromies around $r = 0$ and $r = 1$. Solution of this equation again leads to the quasinormal mode frequencies. In both the approaches mentioned above, appropriate boundary conditions are imposed both at the horizon and at infinity.

In a recent development, Birmingham and Carlip [9] have analyzed the BTZ black hole perturbations using boundary conditions formulated in terms of relation between monodromies at the inner and outer horizons. Their treatment does not refer to any boundary condition at infinity and no such condition is imposed upon the solutions of the wave equation. These monodromy conditions however give rise to the usual quasinormal modes of the BTZ black hole and the standard ADS/CFT correspondence for the BTZ black hole [10] holds for the modes derived from these relations as well. The application of these monodromy relations to higher dimensional near-extremal black holes with asymptotically flat geometries, whose near-horizon region contains the geometry of a BTZ black hole, gives rise to the so called "non-quasinormal modes", as the latter are obtained without imposing any boundary condition at infinity. As emphasized in [9], the conditions on the monodromies in general do not follow from the usual boundary conditions of the system, but are somewhat ad hoc.

In this Letter we shall show that the results obtained in Ref. [9] follow directly from the properties of the BTZ black hole and the monodromy conditions assumed there can be justified. The basic feature of the BTZ black hole which allows this to happen is that the Euclidean BTZ is a hyperbolic 3-manifold which is geometrically finite [12]. The three dimensional hyperbolic structures for such a manifold, according to Sullivan's theorem [11], are in 1-1 correspondence with the two dimensional conformal structures of its boundary. This is nothing but a precise mathematical statement of holography for the BTZ black hole [12, 13]. For the Euclidean BTZ black hole, the two dimensional conformal structure of its boundary is described by the Teichmuller parameter τ of a two torus T^2. Two such parameters τ and τ' are considered equivalent

if they are related by the action of the modular group. On the other hand, the monodromies of the solutions of the wave equation at inner and outer horizon are determined by the hyperbolic structures encoded in the metric of the BTZ manifold. Thus, the 1-1 correspondence between the conformal structures and the hyperbolic structures following from Sullivan's theorem requires that the monodromies be invariant under the action of the modular group as well. We shall show that imposing this requirement leads to the non-quasinormal modes of Birmingham and Carlip [9]. We start with a brief summary of the properties of the BTZ black hole that is relevant for our purposes.

We shall first review the hyperbolic structure of the BTZ black hole. The metric of a BTZ black hole with mass M and angular momentum J is given by

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J}{2r^2} dt\right)^2 \tag{1}$$

where $\Lambda = -1l^2$ is the cosmological constant and the Newton's constant G satisfies the condition $8G = 1$. The inner and outer horizons are given $r = \rho_+$ and $r = \rho_-$ respectively where

$$\rho_\pm = \left[\frac{Ml^2}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}}\right)\right]^{\frac{1}{2}}. \tag{2}$$

For our purpose it is necessary to consider the Euclidean continuation of the BTZ black hole, which is obtained from (1) by setting $t = it_E$ and $J = -iJ_E$. The Euclidean metric is then given by

$$ds_E^2 = \left(-M + \frac{r^2}{l^2} - \frac{J_E^2}{4r^2}\right) dt_E^2 + \left(-M + \frac{r^2}{l^2} - \frac{J_E^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J_E}{2r^2} dt_E\right)^2. \tag{3}$$

In the Euclidean case the inner and outer horizons are given by

$$r_\pm = \left[\frac{Ml^2}{2}\left(1 \pm \sqrt{1 + \frac{J_E^2}{M^2 l^2}}\right)\right]^{\frac{1}{2}}. \tag{4}$$

In terms of the variables

$$\begin{aligned}
x &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \cos\left(\frac{r_+}{l^2} t_E + \frac{|r_-|}{l}\phi\right) \exp\left(\frac{r_+}{l}\phi - \frac{|r_-|}{l^2} t_E\right) \\
y &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \sin\left(\frac{r_+}{l^2} t_E + \frac{|r_-|}{l}\phi\right) \exp\left(\frac{r_+}{l}\phi - \frac{|r_-|}{l^2} t_E\right) \\
z &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{\frac{1}{2}} \exp\left(\frac{r_+}{l}\phi - \frac{|r_-|}{l^2} t_E\right)
\end{aligned} \tag{5}$$

the Euclidean metric (3) becomes

$$ds_E^2 = \frac{l^2}{z^2}(dx^2 + dy^2 + dz^2), \quad z > 0. \tag{6}$$

The metric in (6) is that of the upper half space of the the three dimensional hyperbolic space H^3 and thus BTZ is locally isometric to H^3. However, the periodicity of the coordinate ϕ leads to the identifications

$$(x,y,z) \sim e^{\frac{2\pi r_+}{l}} \left(x \cos\left(\frac{2\pi|r_-|}{l}\right) - y \sin\left(\frac{2\pi|r_-|}{l}\right), x \sin\left(\frac{2\pi|r_-|}{l}\right) + y \cos\left(\frac{2\pi|r_-|}{l}\right), z \right). \tag{7}$$

In terms of the spherical polar coordinates in the upper half plane given by

$$(x,y,z) = (R \cos\theta \cos\xi,\ R \sin\theta \cos\xi,\ R \sin\xi), \tag{8}$$

we can write

$$ds_E^2 = \frac{l^2}{\sin^2\xi} \left(\frac{dR^2}{R^2} + d\xi^2 + \cos^\xi d\theta^2 \right), \tag{9}$$

with the identifications in (7) now being given by

$$(R,\theta,\xi) \sim \left(Re^{\frac{2\pi r_+}{l}},\ \theta + \frac{2\pi|r_-|}{l},\ \xi \right). \tag{10}$$

The resulting manifold has topology of a solid torus, each constant $\xi \neq \frac{\pi}{2}$ section of which is a two torus T^2 [14].

In order to proceed, the next point to note is that the BTZ black hole, which is locally isomorphic to H^3 is geometrically finite [12]. This feature establishes a geometric result that is equivalent to a precise mathematical statement of holography. It establishes, by using a theorem (Sullivan's theorem) [11], the equivalence of the hyperbolic structures of the BTZ 3-manifold with the conformal structures of its boundary. More precisely, if K is a geometrically finite hyperbolic 3-manifold with boundary then Sullivan's theorem states that as long as K admits one hyperbolic realization, there is a 1-1 correspondence between hyperbolic structures on K and conformal structures on its boundary ∂K, the latter being the Teichmuller space of the boundary ∂K.

Let us now analyze the implication of Sullivan's theorem for the BTZ black hole. As discussed above, the boundary of the BTZ black hole has the topology of T^2 and the corresponding Teichmuller space is given by a the fundamental region of the complex variable τ. In other words, two Teichmuller parameters τ and τ' are equivalent if

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1 \tag{11}$$

and $a,b,c,d \in Z$. The transformation in (11) is nothing but the action of the modular group on τ generated by the operations

$$\begin{aligned} S &: \tau \to -\frac{1}{\tau} \\ T &: \tau \to \tau + 1. \end{aligned} \tag{12}$$

In terms of the horizon radii of the Euclidean BTZ black hole, the effective action of the modular group can be written as

$$\begin{aligned} S &: r_+ \leftrightarrow r_- \\ T &: r_+ \to r_+, \quad r_- \to r_- + r_+. \end{aligned} \tag{13}$$

By analytically continuing back to the corresponding Minkowskian variables, S and T would have analogous actions on $\rho_\pm$ respectively.

According to Sullivan's theorem, the conformal structures at the boundary of the BTZ black hole are in 1-1 correspondence with the hyperbolic structures encoded in the BTZ metric. It should thus be possible to have a corresponding action of the modular group on the quantities determined by the hyperbolic structure, i.e. the metric of the BTZ black hole. In this case, such a natural object determined by the metric is given by the monodromy of the solution of the wave equation in the BTZ background. Below we briefly recall how such monodromies arise in this context.

Following Birmingham and Carlip [6, 9], we consider the wave equation for a massless scalar field χ in the background of the BTZ black hole given by

$$\bigtriangledown^2 \chi = 0. \tag{14}$$

Using the separation of variables

$$\chi = R(r)\ \mathrm{e}^{-iwt}\ \mathrm{e}^{im\phi}, \tag{15}$$

the equation satisfied by the function $R(r)$ can be written in the form of the standard hypergeometric equation [6]. Near $r = \rho_+$, the radial part of the ingoing solution is given by

$$R(\rho_+) \sim (r^2 - \rho_+^2)^{-\frac{i}{8\pi}[\beta_R(w+\frac{m}{l})\ +\ \beta_L(w-\frac{m}{l})]}, \tag{16}$$

where $\beta_{R,L} = \frac{2\pi l^2}{(\rho_+ \pm \rho_-)}$. The monodromy of this solution is given by its change under 2π rotation around $r = \rho_+$ and can be written as

$$P(\rho_+) = \exp\left(\frac{1}{4}\left[\beta_R\left(w + \frac{m}{l}\right) + \beta_L\left(w - \frac{m}{l}\right)\right]\right). \tag{17}$$

The above solution can be analytically continued and near $r = \rho_-$, it can be expressed as a suitable linear combination of two functions

$$R^\pm(\rho_-) \sim (r^2 - \rho_-^2)^{\pm\frac{1}{8\pi}[\beta_R(w+\frac{m}{l})\ -\ \beta_L(w-\frac{m}{l})]}. \tag{18}$$

The monodromies of these solutions at $r = \rho_-$ are given by

$$P^\pm(\rho_-) = \exp\left(\pm\frac{1}{4}\left[\beta_R\left(w + \frac{m}{l}\right) - \beta_L\left(w - \frac{m}{l}\right)\right]\right). \tag{19}$$

At this stage, Birmingham and Carlip [9] imposed the following conditions on the monodromies at the inner and outer horizons:

$$P(\rho_+)\ P(\rho_-) = 1. \tag{20}$$

It was also shown by them that this condition is satisfied if either

$$P(\rho_+)\ P^+(\rho_-) = 1 \tag{21}$$

or

$$P(\rho_+)\ P^-(\rho_-) = 1. \tag{22}$$

These conditions on monodromies are clearly different from the usual Dirichlet boundary condition imposed at infinity. Due to this reason the frequencies obtained from the above equations are called the non-quasinormal mode frequencies and are given by [9]

$$w_L = \frac{m_L}{l} - \frac{4\pi i}{\beta_L} n \tag{23}$$

and

$$w_R = \frac{m_R}{l} - \frac{4\pi i}{\beta_R} n \tag{24}$$

where $n \in Z$ and $m_L,\ m_R \in Z$. It may be noted that the AdS/CFT correspondence discussed in ref. [10] is consistent with these non-quasinormal mode frequencies as well.

Our goal now is to provide a justification for the eqns. (20) - (22) using the geometric finiteness property of the BTZ black hole. As argued before, Sullivan's theorem in this case leads to a natural action of the modular group on the monodromies. Since the Teichmuller parameters related by the action of the modular group are considered equivalent, Sullivan's theorem therefore suggests that the monodromies should be left invariant under the action of the modular group.

Let us now analyze the implication of the above assertion. Consider the action of the generator S of the modular group on the monodromy $P(\rho_+)$. As a result of our assertion, we demand that

$$SP(\rho_+) = P(\rho_+), \tag{25}$$

i.e. the monodromy is kept invariant under the action of the modular group. However, using (13) we get

$$SP(\rho_+) = P(S\rho_+) = P(\rho_-) \tag{26}$$

However, for $P(\rho_-)$, we could take either $P^+(\rho_-)$ or $P^-(\rho_-)$. Therefore, the invariance of the monodromies under the action of the modular group leads to the equations

$$P(\rho_+) = P^+(\rho_-) \tag{27}$$

or

$$P(\rho_+) = P^-(\rho_-), \tag{28}$$

which are the same as imposed in ref. [9] and they lead to the same non-quasinormal mode frequencies as given in (23) and (24). Using the above values of w_L and w_R it is also easy to see that the action of T on the monodromies do not lead to any new condition.

We have therefore shown that the conditions on the monodromies used to evaluate the frequencies of the non-quasinormal modes introduced in ref. [9] follow from the geometric properties of the BTZ black hole. Our analysis is based on the observation made in [12] that BTZ black hole is geometrically finite and thus Sullivan's theorem is applicable in this case. Using Sullivan's theorem, the hyperbolic structures of the BTZ black hole are related to Teichmuller parameters of its boundary, which is T^2. On the Teichmuller space of T^2, there is

a natural action of the modular group and two different Teichmuller parameters related by the action of the modular group are equivalent. As a consequence of Sullivan's theorem, this leads to the condition that the monodromies be invariant under the action of the modular group. This invariance condition on the monodromies immediately leads to the relations between the monodromies that were imposed in ref. [9] in an ad hoc fashion. We are thus able to give a geometric interpretation of the monodromy conditions imposed by Birmingham and Carlip. Moreover, this geometric condition is strongly related to the holographic nature of the BTZ black hole. It may also be noted that a variety of black holes arising from string theory have a near-horizon geometry containing the BTZ black hole. It is thus plausible that our analysis applies to this large class of black holes as well.

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